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Fluctuations in radiation scattered into the Fresnel region by a random-phase screen in uniform motion

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Abstract. The scattering of radiation by a rigid deep random-phase screen in uniform linear motion is studied and formulae are derived for the spatial and temporal correlation functions of the scattered field and intensity in the Fraunhöfer and Fresnel regions. A joint Gaussian distribution is used to represent the phase screen and the illuminating beam is assumed to have a curved wavefront and Gaussian intensity profile. It is shown that the coherence properties of the scattered radiation depend on an apparent area of illumination which is a function both of the actual width of the illuminating beam and of the slope distribution of the scattered wavefront. The dependence of the second intensity moment on distance from the screen is discussed and compared with that predicted by previous authors.

1. Introduction

A random-phase screen is a thin scattering layer which introduces randomly fluctuating path differences into an incident electromagnetic wave. The properties of radiation scattered by such a system have been investigated by many authors since the pioneering work of Booker *et al* (1950) and Ratcliffe (1956) because of their importance in connection with the scintillation of extra-terrestial radio sources caused by irregularities in the ionosphere or in the solar wind (for example Little and Lovell 1950, Hewish *et al* 1964). Significant theoretical contributions to the subject have been made by Mercier (1962), Salpeter (1967), Bramley and Young (1967) and more recently by Jokipii (1970), Whale (1973, 1974) and by Taylor (1972) and Taylor and Infosino (1975). Useful reviews of some of the earlier work were given by Briggs (1966) and Singleton (1970). These authors investigated the problem of scattering of an incident plane wave by an infinite one- or two-dimensional screen introducing Gaussian distributed, random-phase fluctuations, and examined the statistical and spatial coherence properties of the scattered radiation as a function of distance from the screen. Several interesting features emerged from the analyses. For example, in the deep phase-screen limit when path differences are introduced which exceed about one-third of the wavelength of the incident radiation, it was found that the variance of the scattered intensity (square of the envelope of the field) increased from zero near the screen to a maximum in the region of focusing predicted by geometrical optics and then decreased with distance, finally saturating at a value of unity far from the scattering plane.

Much of the above mentioned work was motivated by the possibility of using measurements of radio-star scintillation to deduce properties of the earth's ionosphere. More recently, a similar desire to determine the properties of a phase screen from measurements of fluctuations in scattered electromagnetic radiation has led to a theoretical investigation of a somewhat different scattering configuration. In 1969, Deutsch and Keating showed that under certain conditions a thin (25 μm) layer of nematic liquid crystal undergoing electrically driven hydrodynamic turbulence introduced phase fluctuations into an incident optical frequency field. Subsequently it was shown (Jakeman and Raynes 1972, Jakeman and Pusey 1973b, Pusey and Jakeman 1975) that the system in fact approximated a deep random-phase screen when illuminated with radiation from a 6328 Å He–Ne laser. Laser light scattering experiments on this type of system are typically carried out by measuring intensity fluctuations in the *Fraunhöfer region* where they are Gaussian distributed (corresponding to the saturation region mentioned above where the variance is unity) and contain no information regarding the correlation length of the phase fluctuations. In order to develop a technique to obtain this kind of information from Fraunhöfer region observations, therefore, a theoretical investigation was made on the effect of reducing the illuminated area by focusing down the laser beam to a small spot on the scatterer, assumed to be a deep, Gaussian random-phase screen (Jakeman and Pusey 1973a, Jakeman 1974, Jakeman and Pusey 1975). In such a situation the probability distribution of the scattered field is non-Gaussian and the statistical properties of the intensity fluctuations can be related to the parameters of an assumed model for the phase fluctuations. This configuration may be contrasted with the problem studied previously (e.g. Mercier 1962) in which the phase screen was assumed to be infinite in lateral extent with non-Gaussian fluctuations arising from focusing effects occurring in the *Fresnel region* with respect to the scattering plane.

In addition to calculating the statistical and spatial coherence properties of light scattered by a restricted area of a deep random-phase screen, Jakeman and Pusey (1975) investigated its temporal coherence properties which do not seem to have received much attention from earlier workers in the field but which were of interest in the case of the liquid crystal system. It was pointed out, however, that although the analysis was valid for scattering from an intrinsically fluctuating phase screen, it could not be applied to translating 'rigid' systems such as, for example, a moving or scanned perfectly conducting rough surface. This type of scatterer may provide one of the most important applications of phase screen theory, at least at optical frequencies. The problem of scattering from rough surfaces is of course a long standing one (see for example Beckmann and Spizzichino 1963 and references therein) but since the advent of the laser there has been renewed interest in the subject. Early work on the scattering of coherent light from surfaces such as ground glass was mainly concerned with the fundamental properties of detected optical signals (for example, Martienssen and Spiller 1964, Arecchi 1965) but a good deal of effort has been devoted recently to the application of the knowledge so gained to practical problems such as the measurement of translational and vibrational motion of solid surfaces (for a review, see Birch 1975) and surface roughness (Sprague 1972, Nagata *et al* 1973, Fujii and Asakura 1974, George and Jain 1974, Parry 1974a, b, Pedersen 1974, 1975, Takai 1975, Leger *et al* 1975, Ohtsubo and Asakura 1975). Most theoretical calculations made in this connection have been based on approximations valid in the Fraunhöfer region and assume that the incident wavefront and surface are planar. However, it has been known for many years that several commonly observed phenomena are due to curvature effects arising either due to divergence or convergence of the incident radiation, curvature of the illuminated surface or observation in the Fresnel region (for example Rigden and Gordon 1962, Oliver 1963, Sporton 1969). Several papers have appeared on the subject recently in which attempts have been made to quantify the various effects of wavefront curvature (Estes *et al* 1971, Nagata and Umehara 1973, Takai 1974, Jakeman 1975) but calculations appear to have been confined, hitherto, to special limiting cases (for example when the scattered field is Gaussian distributed) and no attempt has been made to extend the theoretical analyses to predict intensity statistics or coherence properties in non-Gaussian situations (arising either from focusing or restrictions on the illuminated area) when the results become sensitive to surface detail.

In this paper, therefore, we extend the deep phase-screen calculations of Jakeman and Pusey (1975) to include the effect of curvature (so that the results are valid in the Fresnel region) and to cope with the case of a uniformly moving rigid rough surface. Insofar as the approximations used are valid the work generalizes some of the deep phase-screen results of workers in the fields of radio astronomy and ionospheric physics to include the effects of both uniform translation of the phase screen and of spatial limitation of the observed scattering region.

The theoretical approach used in earlier papers is briefly reviewed in § 2 and extended to include the effect of a uniform translation of the phase screen. Results for the first- and second-order coherence properties of the scattered radiation are presented and some special cases considered. Section 3 is devoted to a discussion of the results and their interpretation, and comparisons are made with earlier work. The main conclusions are summarized in § 4.

2. Theory

Consider the experimental set up shown in figure 1. A beam of monochromatic electromagnetic radiation with a Gaussian profile of width W and radius of curvature σ is incident on a rigid phase screen of negligible thickness which is moving with velocity $\boldsymbol{v}$ perpendicular to the beam. The forward scattered radiation is detected by a square-law envelope detector whose axis makes an angle θ_1 with the direction of incidence.

After passing through the phase screen the positive frequency part of the electric field may be written

$$\mathscr{E}^{+}(\boldsymbol{r}, 0; t) = E_0 \exp(-\mathrm{i}\omega t) \exp\left(\frac{\mathrm{i}kr^2}{2\sigma}\right) \exp(\mathrm{i}\phi(\boldsymbol{r}; t)) \exp(-r^2/W^2) \qquad (1)$$

where $k=\omega/c$ is the wavenumber of the radiation, $\phi(\boldsymbol{r}; t)$ is the randomly varying position-dependent phase shift caused by transmission through the moving screen and E_0 is a constant.

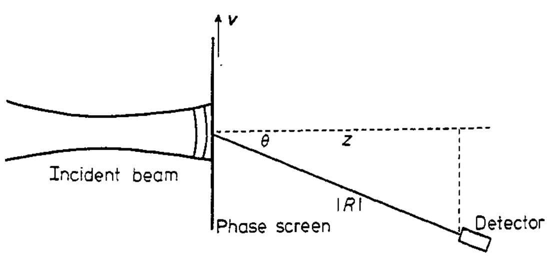

Figure 1. Scattering geometry.

In accordance with the Helmholtz formula the positive frequency part of the field at the point $\boldsymbol{R}_1 = (\boldsymbol{r}_1, z)$ is given by

$$\mathscr{E}^+(\boldsymbol{R}_1; t) = \frac{\mathrm{i}E_0(1+\cos\theta_1)}{2\lambda R}\exp(-\mathrm{i}\omega t)\int_{-\infty}^{\infty} \mathrm{d}^2\boldsymbol{r}'\exp(\mathrm{i}k|\boldsymbol{R}_1-\boldsymbol{r}'|)$$

$$\times\exp\left(\frac{\mathrm{i}kr'^2}{2\sigma}\right)\exp(\mathrm{i}\phi(\boldsymbol{r}'; t))\exp(-r'^2/W^2) \tag{2}$$

where $R = |\boldsymbol{R}_1|$ and λ is the wavelength of the radiation. Then, assuming that θ_1 is small so that $r_1 \ll R$, it follows from the theory of Fresnel diffraction that

$$|\boldsymbol{r}'-\boldsymbol{R}_1| \simeq R - \frac{\boldsymbol{r}'\cdot\boldsymbol{R}_1}{R} + \frac{r'^2}{2R} \tag{3}$$

and so

$$\mathscr{E}^+(\boldsymbol{R}_1; t) = \frac{\mathrm{i}E_0(1+\cos\theta_1)}{2\lambda R}\exp[\mathrm{i}(kR-\omega t)]$$

$$\times\int_{-\infty}^{\infty} \mathrm{d}^2\boldsymbol{r}'\exp[\mathrm{i}k\kappa r'^2 - \mathrm{i}k\boldsymbol{r}'\cdot\boldsymbol{r}_1/R + \mathrm{i}\phi(\boldsymbol{r}'; t) - r'^2/W^2] \tag{4}$$

where

$$\kappa = \frac{1}{2}\left(\frac{1}{\sigma}+\frac{1}{R}\right). \tag{5}$$

Assuming that ϕ is Gaussian distributed so that

$$\left\langle\exp\left(-\mathrm{i}\sum_j \phi(\boldsymbol{r}_j; t)\right)\right\rangle = \exp\left[-\tfrac{1}{2}\left\langle\left(\sum_j \phi(\boldsymbol{r}_j; t)\right)^2\right\rangle\right] \tag{6}$$

the space–time field correlation function may be written, after some manipulation, in the form

$$\langle\mathscr{E}^+(\boldsymbol{R}_1; t)\mathscr{E}^-(\boldsymbol{R}_2; t+\tau)\rangle$$

$$= \frac{|E_0|^2(1+\cos\theta_1)(1+\cos\theta_2)}{16\lambda^2R^2}\exp(\mathrm{i}\omega\tau)\exp(-\overline{\phi^2})\int_{-\infty}^{\infty}\mathrm{d}^2\boldsymbol{t}'\,\mathrm{d}^2\boldsymbol{t}''$$

$$\times\exp[\tfrac{1}{2}\mathrm{i}k(\boldsymbol{t}'\cdot\boldsymbol{U}+\boldsymbol{t}''\cdot\boldsymbol{V})]\exp\left(-\frac{(t'^2+t''^2)}{W^2}\right)$$

$$\times\exp(\mathrm{i}k\kappa\boldsymbol{t}'\cdot\boldsymbol{t}'')\exp(-\overline{\phi^2}\rho(\boldsymbol{t}'+\boldsymbol{v}\tau)) \tag{7}$$

where the detection points have been chosen, for convenience, such that $|\boldsymbol{R}_1| = |\boldsymbol{R}_2| = R$ and we have defined

$$\boldsymbol{U} = \frac{1}{R}(\boldsymbol{r}_1+\boldsymbol{r}_2) \qquad \boldsymbol{V} = \frac{1}{R}(\boldsymbol{r}_1-\boldsymbol{r}_2). \tag{8}$$

In deriving equation (7) we have made use of the fact that for a rigid phase screen translating with velocity $\boldsymbol{v}$

$$\phi(\boldsymbol{r}; t+\tau) = \phi(\boldsymbol{r}-\boldsymbol{v}\tau; t) \tag{9}$$

and so the normalized phase correlation function, assumed to be translationally invariant and stationary, is given by

$$\langle\phi(\boldsymbol{r}'; t)\phi(\boldsymbol{r}''; t+\tau)\rangle/\overline{\phi^2} = \langle\phi(\boldsymbol{r}'; t)\phi(\boldsymbol{r}''-\boldsymbol{v}\tau; t)\rangle/\overline{\phi^2} = \rho(\boldsymbol{r}'-\boldsymbol{r}''+\boldsymbol{v}\tau) \tag{10}$$

where $\overline{\phi^2}$ is the mean square phase deviation. Similarly the space–time intensity correlation function may be written in the form

$$\langle I(\boldsymbol{R}_1; t) I(\boldsymbol{R}_2; t+\tau)\rangle$$
$$= \frac{\pi W^2 |E_0|^4 (1+\cos\theta_1)^2 (1+\cos\theta_2)^2}{16\lambda^4 R^4} \exp(-2\overline{\phi^2})$$
$$\times \int_{-\infty}^{\infty} d^2 t'\, d^2 t''\, d^2 t''' \exp[-ik(\boldsymbol{t}''.\boldsymbol{U} + \boldsymbol{t}'''.\boldsymbol{V})] \exp(2ik\kappa \boldsymbol{t}'''.\boldsymbol{t}')$$
$$\times \exp\{-[t'^2 + t'''^2 + t''^2(1+k^2\kappa^2 W^4)]/W^2\}$$
$$\times \exp\{\overline{\phi^2}[\rho(\boldsymbol{t}''+\boldsymbol{t}''') + \rho(\boldsymbol{t}''-\boldsymbol{t}''') + \rho(\boldsymbol{t}'+\boldsymbol{t}''+\boldsymbol{v}\tau) + \rho(\boldsymbol{t}'-\boldsymbol{t}''+\boldsymbol{v}\tau)$$
$$-\rho(\boldsymbol{t}'+\boldsymbol{t}'''+\boldsymbol{v}\tau) - \rho(\boldsymbol{t}'-\boldsymbol{t}'''+\boldsymbol{v}\tau)]\} \quad (11)$$

where

$$I(\boldsymbol{R}, t) = \mathscr{E}^+(\boldsymbol{R}; t)\mathscr{E}^-(\boldsymbol{R}; t). \quad (12)$$

For large $\overline{\phi^2}$, equation (7) may be evaluated using the method of steepest descent. In order to evaluate the integrals in equation (11), however, we introduce the approximation

$$\exp[\overline{\phi^2}\rho(\boldsymbol{r})] = 1 + [\exp(\overline{\phi^2}) - 1]\exp(-\overline{\phi^2} r^2/\xi^2) \quad (13)$$

used in earlier work (e.g. Berry 1973, Jakeman 1974), assuming that

$$\overline{\phi^2} \gg 1 \quad (14)$$

and that the phase correlation function $\rho(\boldsymbol{r})$ may be expanded about the origin in terms of a 'correlation length' ξ in the form

$$\rho(\boldsymbol{r}) = 1 - r^2/\xi^2 + \dots. \quad (15)$$

The use of (13) and further simplifications to evaluate the second-order correlation function (11) has been shown to be equivalent to taking a 'facet' model for the scattered wavefront (Jakeman and Pusey 1975) and strictly speaking the results obtained for this quantity are valid only for such scattering systems. However this restriction does not apply when many correlation areas of the scatterer contribute to the field (the Gaussian limit) nor does it apply to the first-order correlation function (7) which is evaluated using (14) and (15) to give

$$g^{(1)}(\boldsymbol{R}_1, \boldsymbol{R}_2; \tau)$$
$$= \langle \mathscr{E}^+(\boldsymbol{R}_1; t)\mathscr{E}^-(\boldsymbol{R}_2; t+\tau)\rangle / \{\langle I(\boldsymbol{R}_1; t)\rangle\langle I(\boldsymbol{R}_2; t)\rangle\}^{1/2}$$
$$= \exp(i\omega\tau)\exp\left(\frac{ikW_\kappa^2 \boldsymbol{U}.\boldsymbol{v}\tau}{2W^2}\right)\exp\left(\frac{ik^2\xi^2\kappa W_\kappa^2 \boldsymbol{V}.\boldsymbol{U}}{8\overline{\phi^2}}\right)\exp\left(-\frac{v^2\tau^2 W_\kappa^2}{2W^4}\right)$$
$$\times \exp(-\tfrac{1}{2}k^2 W_\kappa^2 |\tfrac{1}{2}\boldsymbol{V} + \kappa \boldsymbol{v}\tau|^2) \quad (16)$$

where, in accordance with the deep phase-screen approximation, terms of order $\exp(-\overline{\phi^2})$ have been neglected and it is further assumed that ξ^2/W_κ^2 is not much greater than one, such that

$$\xi^2/W_\kappa^2 \ll \overline{\phi^2}. \quad (17)$$

It is convenient to express the above result in terms of an apparent beam width W_κ defined by

$$\frac{1}{W_\kappa^2}=\frac{1}{W^2}+\frac{k^2\kappa^2\xi^2}{2\overline{\phi^2}}. \tag{18}$$

Takai (1974) has recently attempted a calculation in this limit for $\boldsymbol{V}=0$ but does not seem to have retained the second curvature dependent term in equation (18).

In evaluating the intensity correlation function the approximation (13) is substituted into equation (11) and leads to a sum of sixteen terms each of which is a ratio of two factors and several of which yield contributions of order $\exp(-\overline{\phi^2})$ in the result and may be neglected. Three major terms are retained in our model, the term

$$\exp(-2\overline{\phi^2})\exp\{\overline{\phi^2}[\rho(\boldsymbol{t}''+\boldsymbol{t}''')+\rho(\boldsymbol{t}''-\boldsymbol{t}''')+\ldots]\}$$

in equation (11) being replaced by

$$\begin{aligned}\Big[&\exp\Big(-\frac{\overline{\phi^2}}{\xi^2}(|\boldsymbol{t}''+\boldsymbol{t}'''|^2+|\boldsymbol{t}''-\boldsymbol{t}'''|^2)\Big)+\exp\Big(-\frac{\overline{\phi^2}}{\xi^2}(|\boldsymbol{t}'-\boldsymbol{t}''+\boldsymbol{v}\tau|^2+|\boldsymbol{t}'+\boldsymbol{t}''+\boldsymbol{v}\tau|^2)\Big)\\ &+\exp(2\overline{\phi^2})\exp\Big(-\frac{\overline{\phi^2}}{\xi^2}(|\boldsymbol{t}''+\boldsymbol{t}'''|^2+|\boldsymbol{t}''-\boldsymbol{t}'''|^2+|\boldsymbol{t}'+\boldsymbol{t}''+\boldsymbol{v}\tau|^2\\ &+|\boldsymbol{t}'-\boldsymbol{t}''+\boldsymbol{v}\tau|^2)\Big)\Big]\Big\{\Big[1+[\exp(\overline{\phi^2})-1]\exp\Big(-\frac{\overline{\phi^2}}{\xi^2}|\boldsymbol{t}'+\boldsymbol{t}'''+\boldsymbol{v}\tau|^2\Big)\Big]\\ &\times\Big[1+[\exp(\overline{\phi^2})-1]\exp\Big(-\frac{\overline{\phi^2}}{\xi^2}|\boldsymbol{t}'-\boldsymbol{t}'''+\boldsymbol{v}\tau|^2\Big)\Big]\Big\}^{-1}.\end{aligned} \tag{19}$$

It is interesting to note that if the surface is replaced by a finite number of scattering points $\boldsymbol{t}_i$ imparting random phase values ϕ_i which are uncorrelated so that

$$\begin{aligned}\rho(\boldsymbol{t}_i-\boldsymbol{t}_j)&=1 \qquad \text{if } \boldsymbol{t}_i=\boldsymbol{t}_j\\ &=0 \qquad \text{otherwise}\end{aligned} \tag{20}$$

then the relationship

$$\exp[\overline{\phi^2}\rho(\boldsymbol{t}_i-\boldsymbol{t}_j)]=1+[\exp(\overline{\phi^2})-1]\rho(\boldsymbol{t}_i-\boldsymbol{t}_j) \tag{21}$$

holds *exactly* and it is easily shown that with respect to the integral in (11) and to order $\exp(-\overline{\phi^2})$

$$\begin{aligned}\exp(-2\overline{\phi^2})&\exp\{\overline{\phi^2}[\rho(\boldsymbol{t}_j+\boldsymbol{t}_k)+\rho(\boldsymbol{t}_j-\boldsymbol{t}_k)+\rho(\boldsymbol{t}_i+\boldsymbol{t}_j+\boldsymbol{v}\tau)+\rho(\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{v}\tau)]\}\\ &\equiv\rho(\boldsymbol{t}_j+\boldsymbol{t}_k)\rho(\boldsymbol{t}_j-\boldsymbol{t}_k)+\rho(\boldsymbol{t}_i+\boldsymbol{t}_j+\boldsymbol{v}\tau)\rho(\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{v}\tau)\\ &\quad+\exp(2\overline{\phi^2})\rho(\boldsymbol{t}_j+\boldsymbol{t}_k)\rho(\boldsymbol{t}_j-\boldsymbol{t}_k)\rho(\boldsymbol{t}_i+\boldsymbol{t}_j+\boldsymbol{v}\tau)\rho(\boldsymbol{t}_i-\boldsymbol{t}_j+\boldsymbol{v}\tau).\end{aligned} \tag{22}$$

In the discrete case, the three terms which contribute to the intensity correlation function define the situations

$$\boldsymbol{t}_j=\boldsymbol{t}_k=0; \qquad \boldsymbol{t}_j=0,\quad \boldsymbol{t}_i=-\boldsymbol{v}\tau; \qquad \boldsymbol{t}_j=\boldsymbol{t}_k=0,\quad \boldsymbol{t}_i=-\boldsymbol{v}\tau \tag{23}$$

and correspond for a continuum phase screen to those terms which have been retained in equation (19)

Evaluation of the integrals in equation (11) is outlined in the appendix. Simplification of the results occurs in the following special cases:

(1) In the Gaussian limit $\xi^2/W_\kappa^2 \to 0$, when there is a large number of phase 'coherence areas' within the apparent beam width, equation (A.8) factorizes (for example Siegert (1943)) to become

$$g^{(2)}(\boldsymbol{R}_1, \boldsymbol{R}_2; \tau) = 1 + |g^{(1)}(\boldsymbol{R}_1, \boldsymbol{R}_2; \tau)|^2 \tag{24}$$

with $g^{(1)}$ given by equation (16) above.

(2) Setting $\boldsymbol{V}$ and $\tau = 0$ in equation (A.8) yields the second-order statistic

$$\frac{\langle I^2(\boldsymbol{R}; t)\rangle}{\langle I(\boldsymbol{R}; t)\rangle^2} = 2\exp\left(-\frac{\xi^2}{W_\kappa^2}\right) + \frac{2\overline{\phi^2} W^6}{\xi^2 W_\kappa^2 (W^2 + W_\kappa^2)(1 + k^2\kappa^2 W^4)} \exp\left(\frac{k^2\xi^2 W_\kappa^4 \sin^2\theta}{2\overline{\phi^2} W^2 (W^2 + W_\kappa^2)}\right)$$
$$\times \left\{\left[\cos\frac{k\kappa\xi^2}{2} - \frac{\pi k\kappa\xi^2}{2\overline{\phi^2}} \exp\left(-\frac{\xi^2}{2W^2}\right) \operatorname{cosech}\frac{\pi k\kappa\xi^2}{2\overline{\phi^2}}\right]^2\right.$$
$$\left. + \left[\sin\frac{k\kappa\xi^2}{2} + \frac{\pi\xi^2}{2W^2\overline{\phi^2}} \exp\left(-\frac{\xi^2}{2W^2}\right)\left(\operatorname{cosech}\frac{\pi k\kappa\xi^2}{2\overline{\phi^2}} - \frac{2\overline{\phi^2}}{\pi k\kappa\xi^2}\right)\right]^2\right\}. \tag{25}$$

(3) When $\kappa = 0$ we recover from (25) the result of Jakeman and Pusey (1975)

$$\frac{\langle I^2(\boldsymbol{R}; t)\rangle}{\langle I(\boldsymbol{R}; t)\rangle^2} = 2\exp\left(-\frac{\xi^2}{W^2}\right) + \frac{\overline{\phi^2} W^2}{\xi^2} \exp\left(+\frac{k^2\xi^2 \sin^2\theta}{4\overline{\phi^2}}\right)\left[1 - \exp\left(-\frac{\xi^2}{2W^2}\right)\right]^2. \tag{26}$$

(4) In the limit $W \to \infty$ equation (25) becomes

$$\frac{\langle I^2(\boldsymbol{R}; t)\rangle}{\langle I(\boldsymbol{R}; t)\rangle^2} = 2\exp\left(-\frac{k^2\kappa^2\xi^4}{2\overline{\phi^2}}\right) + \left(\cos\frac{k\kappa\xi^2}{2} - \frac{\pi k\kappa\xi^2}{2\overline{\phi^2}} \operatorname{cosech}\frac{\pi k\kappa\xi^2}{2\overline{\phi^2}}\right)^2 + \sin^2\frac{k\kappa\xi^2}{2}. \tag{27}$$

3. Discussion

The statistical properties embodied in the general results (A.8) and (A.9) are most easily understood in terms of the limiting behaviour of equations (24)–(27). We shall therefore restrict the discussion to consideration of these special cases, concentrating on new features which have not been discussed in our previous publications (Jakeman 1974, Jakeman and Pusey 1975).

In the Gaussian limit given by equations (24) and (16) the intensity correlation function bears a superficial resemblance to a formula obtained earlier by Jakeman (1975). As a function of the detector separation the correlation attains a maximum when $\boldsymbol{V} = -2\kappa\boldsymbol{v}\tau$ and thus the speckle pattern produced by the phase screen is seen to have a curvature-dependent velocity proportional to that of the screen itself. As a function of the delay time τ, $g^{(2)}$ is a shifted Gaussian function of width

$$\tau_c = W^2/vW_\kappa(1 + k^2\kappa^2 W^4)^{1/2} \tag{28}$$

centred at the delay time

$$\tau_d = k^2\kappa W^4 \boldsymbol{V} \cdot \boldsymbol{v}/v^2(1 + k^2\kappa^2 W^4). \tag{29}$$

The last result, which is the time taken for an element of the intensity speckle pattern to traverse the distance separating the detection points, is identical to that obtained previously for scattering by an assembly of uniformly moving point scatterers (Jakeman 1975). However, the speckle fluctuation time, equation (28), is longer than that obtained for the latter system by a factor W/W_κ. This is because (from equations (17) and (18)) W_κ differs significantly from W only when $k\kappa W^2 \gg 1$ and τ_c is then the transit time of a single speckle across the detector (Jakeman 1974). According to equations (24) and (16) the speckle size (obtained by setting $\tau = 0$) is of the order of $2R/kW_\kappa$ and since $W_\kappa < W$ this is larger than that for point scatterers ($2R/kW$) giving an increased fluctuation time for the same overall pattern velocity.

It should be noted that as a result of the definition of equation (18) of W_κ we have the somewhat unusual situation in which the statistics of the scattered field are close to Gaussian but its spatial coherence properties are a function of the structure of the scatterer. In practical terms this situation will occur, for example, when the characteristic tilt of an element of a scattering surface is small compared to the angle of view defined by the radius of the actual illuminated region at the detection point. The relatively simple form of equation (18) is due to the choice of Gaussian functions for both the intensity profile of the incident radiation and the statistics of the phase fluctuations. The κ dependent term arises since radiation falling onto the screen at a distance r from the axis of the beam must be scattered through an angle $2k\kappa r$ to give a return and the probability of slope $\boldsymbol{m}$ in the phase surface may be shown to take the Gaussian form

$$p(\boldsymbol{m}) = \frac{\xi^2}{4\pi\overline{\phi^2}} \exp\left(-\frac{\xi^2 m^2}{4\overline{\phi^2}}\right). \tag{30}$$

In the limit of large W, $W_\kappa \sim (2\overline{\phi^2}/k^2\kappa^2\xi^2)^{1/2}$ which is, roughly speaking, the radius of an area over which the scatterer is capable of giving significant specular returns into the receiving direction.

The concept of an effective illuminated area derives further support from the form of the normalized mean square intensities (equations (26) and (27)). Because of the restriction, equation (14), on $\overline{\phi^2}$, significant deviations from Gaussian statistics can occur even when the illuminated region contains many phase coherence areas, i.e. $\xi^2/W^2 \ll 1$. In this limit equation (26) reduces, in the forward direction, to

$$\frac{\langle I^2(\boldsymbol{R};t)\rangle}{\langle I(\boldsymbol{R};t)\rangle^2} = 2 + \frac{\overline{\phi^2}\xi^2}{4W^2}. \tag{31}$$

On the other hand, sufficiently far from an infinite deep random-phase screen illuminated by plane waves, the second moment of the intensity is given by the small curvature limit $k\kappa\xi^2 \ll 1$ of equation (27)

$$\frac{\langle I^2(\boldsymbol{R},t)\rangle}{\langle I(\boldsymbol{R},t)\rangle^2} = 2 + \frac{\overline{\phi^2}\xi^2}{2W_\kappa^2}. \tag{32}$$

W and W_κ thus clearly play analogous roles in these two limiting situations.

Equation (27) characterizes the situation usually studied by workers in the fields of radio-astronomy and ionospheric physics, i.e. plane waves incident on an infinite Gaussian random-phase screen. It is, of course, valid only for large $\overline{\phi^2}$, but this is just the case in which it has been found difficult, even using numerical techniques, to evaluate the right hand side of equation (11) and hence obtain the second intensity moment. It is interesting, therefore, to compare the result, equation (27), which is based on the approximations of equations (13) and (22), with earlier work. Thus figure 2 compares the second moment given by equation (27) when $\overline{\phi^2} = 10$ with the equivalent curve obtained numerically by Bramley and Young (1967). Agreement is surprisingly good over the range of values of $x = 2/k\kappa\xi^2$ for which equation (27) is

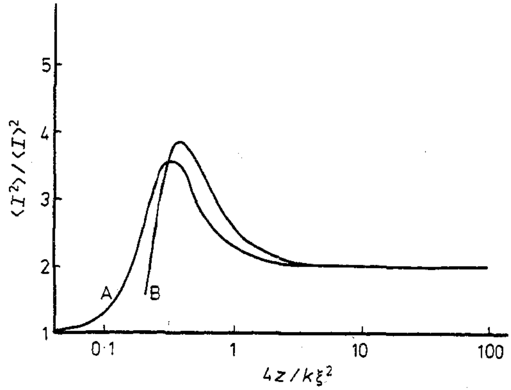

Figure 2. Normalized variance of intensity fluctuations as a function of distance from the phase screen for $\overline{\phi^2} = 10$. (A) Numerically computed result (Bramley and Young 1967). (B) Behaviour predicted by equation (27).

plotted. For small values of this quantity, however, the solution exhibits unphysical oscillations presumably due to the breakdown of the approximations of equations (13) and (22). Earlier work (Jakeman and Pusey 1975) has shown that an analysis based on such approximations is equivalent to a model in which the illuminated area is divided into $N \sim W^2/\xi^2$ specularly reflecting discs of area $\pi\xi^2/2$ whose tilts are Gaussian distributed. Increased fluctuations are then expected (i) as N is reduced and (ii) as the form factors for scattering from individual discs become more directional. In the present situation the first mentioned behaviour is evident in the small curvature limit of equation (32) ($N \sim W_\kappa^2/\xi^2$) which is valid in the Fraunhöfer region with respect to the phase coherence area or single disc. Increased directionality of the radiation pattern scattered by individual discs becomes the dominant source of deviation from Gaussian statistics at larger curvatures, however, and the full formula of equation (27) must then be used to take proper account of Fresnel region effects with respect to these sub-regions of the effective illuminated area. As might be expected from the disc analogy the formula yields a peak near

$$x = 1/\pi \tag{33}$$

corresponding to the 'focus' of such a scatterer. This result may be contrasted with the predictions of other workers based on a geometrical optics approach. For a deep Gaussian phase screen this suggests (Salpeter 1967, Bramley and Young 1967) that focusing should occur close to

$$x = 1/(\overline{\phi^2})^{1/2} \tag{34}$$

which corresponds, incidentally, to $W_\kappa^2 \sim \xi^2$ or $N = 1$. For $\overline{\phi^2} = 10$ there is little difference between equations (33) and (34) so that it is not too surprising that there is good agreement in figure 2 between the peak positions. However, for large $\overline{\phi^2}$ equation (34) predicts that the peak should occur at progressively larger values of the curvature (e.g. closer to the screen) and clearly, in this regime, the model of specularly reflecting discs which is implied by our approximations departs from the situation considered by other authors in which the phase correlation function is assumed to be Gaussian.

It should be noted that non-Gaussian effects in the Fraunhöfer region (due to limitation of the illuminated area) will also be sensitive to the properties of the 'individual scattering centres' contributing to the far field. These properties are embodied in the higher order terms of the expansion equation (15) for the phase correlation function which are attributed very small coefficients by the 'facet' approximation equation (13). The results of a full investigation of this problem will be presented in a future publication.

4. Conclusions

General formulae have been derived for the first- and second-order statistical and coherence properties of radiation scattered by a moving rigid deep random-phase screen illuminated by spherical waves of Gaussian intensity profile. The results are found to depend upon an apparent beam width W_κ which incorporates both the width of the incident beam and also that of the probability distribution of slopes of elements of the phase surface relative to the viewing direction. When this apparent width is large compared to the phase correlation length it is possible to realise a situation in which the statistics of the scattered radiation are nearly Gaussian but the speckle size and intrinsic fluctuation time are determined by W_κ and therefore depend strongly on the structure of the scattering phase screen. The overall motion of the speckle pattern is found to be the same as that predicted previously for point scatterers in these circumstances.

An analytic expression is found for the second moment of the intensity fluctuation distribution as a function of distance from the screen. For a mean square phase deviation of 10 this reproduces the behaviour calculated numerically by Bramley and Young (1967). However, the behaviour close to the screen for very large $\overline{\phi^2}$ is qualitatively different from that predicted by earlier workers in the field due to the different model which is embodied in the approximations made in the present approach.

Appendix

The integral in equation (11) may be written as the sum of three terms T_1, T_2 and T_3 corresponding to the three terms in the numerator of equation (19). Each of the Gaussian terms T_1 and T_2 is evaluated in the same way as the quantity I_1 in the appendix of Jakeman and Pusey (1975) by introducing, respectively, the delta function approximations

$$\exp\left[-t''^2\left(\frac{2\overline{\phi^2}}{\xi^2}+\frac{1}{W^2}\right)\right]\exp[\mathrm{i}\boldsymbol{t}''.(2k\kappa\boldsymbol{t}'-k\boldsymbol{V})]$$

$$\simeq\frac{\pi\xi^2}{2\overline{\phi^2}}\exp\left(-\frac{\xi^2|2k\kappa\boldsymbol{t}'-k\boldsymbol{V}|^2}{8\overline{\phi^2}}\right)\delta^2(\boldsymbol{t}'') \tag{A.1}$$

and

$$\exp\left(-\frac{2\overline{\phi^2}}{\xi^2}|\boldsymbol{t}'+\boldsymbol{v}\tau|^2\right)\exp(-t'^2/W^2)\exp(2\mathrm{i}k\kappa\boldsymbol{t}''.\boldsymbol{t}')$$

$$\simeq\frac{\pi\xi^2}{2\overline{\phi^2}}\exp\left(-\frac{v^2\tau^2}{W^2}\right)\exp(-2\mathrm{i}k\kappa\boldsymbol{t}''.\boldsymbol{v}\tau)\exp\left(-\frac{k^2\kappa^2\xi^2t''^2}{2\overline{\phi^2}}\right)\delta^2(\boldsymbol{t}'+\boldsymbol{v}\tau) \tag{A.2}$$

in accordance with the regions of contribution indicated in equation (23) of this paper. The determination of T_1 and T_2 is thus reduced to evaluating respectively the integrals

$$\int_0^\infty \mathrm{d}s \frac{s\exp(-s^2/W_\kappa^2)I_0[s|(2v\tau/W_\kappa^2)+(k^2\kappa\xi^2\mathbf{V}/2\overline{\phi^2})|]}{\{1+[\exp(\overline{\phi^2})-1]\exp(-\overline{\phi^2}s^2/\xi^2)\}^2} \tag{A.3}$$

and

$$\int_0^\infty \mathrm{d}s \frac{s\exp(-s^2/W_\kappa^2)J_0(s|k\mathbf{V}+2k\kappa v\tau|)}{\{1+[\exp(\overline{\phi^2})-1]\exp(-\overline{\phi^2}s^2/\xi^2)\}^2}. \tag{A.4}$$

The non-Gaussian term T_3 is similar to the quantity I_2 of Jakeman and Pusey (1975) and by transforming to sum and difference coordinates reduces to the evaluation of the quantity II^* where I is

$$\int_0^\infty \mathrm{d}x \frac{x\exp\{-x^2[(1/2W^2)+(\overline{\phi^2}/\xi^2)-(ik\kappa/2)]\}I_0[x|(v\tau/W^2)+\tfrac{1}{2}i(k\mathbf{V}+2k\kappa v\tau)|]}{\{1+[\exp(\overline{\phi^2})-1]\exp(-\overline{\phi^2}x^2/\xi^2)\}} \tag{A.5}$$

and I^* denotes the complex conjugate.

The integrals (A.3), (A.4) and (A.5) are evaluated by splitting the region of integration into two sub-regions from 0–θ and θ–∞, where θ is given by

$$[\exp(\overline{\phi^2})-1]\exp(-\overline{\phi^2}\theta^2/\xi^2)=1 \tag{A.6}$$

so that $\theta^2 \simeq \xi^2$ when $\overline{\phi^2} \gg 1$. In each region the denominator may be expanded binomially and the terms of the series evaluated using the following result

$$\int_0^\theta \mathrm{d}r\, r\exp(-ar^2)J_0(br)$$

$$=\frac{1}{2a}\exp(-a\theta^2)\sum_{t=1}^{\infty}(2a\theta^2)^t\frac{J_t(b\theta)}{(b\theta)^t}$$

$$=\frac{1}{2a}\exp(-b^2/4a)-\frac{1}{2a}\exp(-a\theta^2)\sum_{t=0}^{\infty}\left(\frac{-b^2}{2a}\right)^t\frac{J_t(b\theta)}{(b\theta)^t}. \tag{A.7}$$

In these equations J_t denotes the tth order Bessel function of the first kind but an equivalent result is easily derived in terms of the modified Bessel function I_t. Clearly the first expression on the right hand side of (A.7) converges most rapidly for small values of a, while the second converges most rapidly for large values. Finally, taking into account the inequalities in equations (14) and (17) and assuming that the translation and detector separation are both sufficiently small to ensure that correlation is not damped out by the Gaussian apparent beam width factor, the resulting series may be simplified and the intensity correlation function written in the normalized form.

$$g^{(2)}(\mathbf{R}_1,\mathbf{R}_2;\tau)$$

$$=\langle I(\mathbf{R}_1;t)I(\mathbf{R}_2;t+\tau)\rangle/\langle I(\mathbf{R}_1;t)\rangle\langle I(\mathbf{R}_2;t)\rangle$$

$$\simeq 1-\exp\left(-\frac{\xi^2}{W_\kappa^2}\right)\exp\left(-\frac{W_\kappa^2}{4}\left|\frac{2v\tau}{W_\kappa^2}+\frac{\kappa k^2\xi^2\mathbf{V}}{2\overline{\phi^2}}\right|^2\right)$$

$$\times\sum_{t=1}^{\infty}\left(\frac{2\xi^2}{W_\kappa^2}\right)^t\frac{I_t[\xi|(2v\tau/W_\kappa^2)+(\kappa k^2\xi^2\mathbf{V}/2\overline{\phi^2})|]}{[\xi|(2v\tau/W_\kappa^2)+(\kappa k^2\xi^2\mathbf{V}/2\overline{\phi^2})|]^t}$$

$$+\exp\left(-\frac{v^2\tau^2}{W^2}\right)\exp(-k^2W_\kappa^2|\tfrac{1}{2}\boldsymbol{V}+\kappa\boldsymbol{v}\tau|^2)-\exp\left(-\frac{v^2\tau^2}{W^2}\right)\exp\left(-\frac{\xi^2}{W_\kappa^2}\right)$$
$$\times\sum_{t=1}^{\infty}\left(\frac{2\xi^2}{W_\kappa^2}\right)^t\frac{J_t(2k\xi|\frac{1}{2}\boldsymbol{V}+\kappa\boldsymbol{v}\tau|)}{(2k\xi|\frac{1}{2}\boldsymbol{V}+\kappa\boldsymbol{v}\tau|)^t}$$
$$+\frac{2\overline{\phi^2}\xi^2W^2JJ^*}{W_\kappa^2(W_\kappa^2+W^2)}\exp\left(\frac{k^2U^2\xi^2W_\kappa^4}{8\overline{\phi^2}W^2(W^2+W_\kappa^2)}\right)\exp\left(-\frac{v^2\tau^2}{W^2}\right)\exp\left(-\frac{\xi^2}{W^2}\right) \qquad (A.8)$$

where

$$J=\sum_{t=1}^{\infty}\left(\frac{\xi^2(1-\mathrm{i}k\kappa W^2)}{W^2}\right)^{t-1}\frac{I_t[\xi|(\boldsymbol{v}\tau/W^2)-\mathrm{i}k(\frac{1}{2}\boldsymbol{V}+\kappa\boldsymbol{v}\tau)|]}{[\xi|(\boldsymbol{v}\tau/W^2)-\mathrm{i}k(\frac{1}{2}\boldsymbol{V}+\kappa\boldsymbol{v}\tau)|]^t}$$
$$-\frac{\mathrm{i}\pi}{2\overline{\phi^2}}\left(\operatorname{cosech}\frac{\pi k\kappa\xi^2}{2\overline{\phi^2}}-\frac{2\overline{\phi^2}}{\pi k\kappa\xi^2}\right)I_0[\xi|(\boldsymbol{v}\tau/W^2)-\mathrm{i}k(\tfrac{1}{2}\boldsymbol{V}+\kappa\boldsymbol{v}\tau)|]. \qquad (A.9)$$

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